Energy Localization for Dilatonic Black Holes with a Pure Monopole Field

Irina Radinschi\textsuperscript{1a}, and Th. Grammenos\textsuperscript{2a}

\textsuperscript{1}“Gh. Asachi” Technical University, Iasi, Romania
\textsuperscript{1a}radinschi@yahoo.com,

\textsuperscript{2}University of Thessaly, 383 34 Volos, Greece,
\textsuperscript{2a}thgramme@uth.gr

IWARA09 - 4th International Workshop on Astronomy and Relativistic Astrophysics
October 4 – 8, 2009
Maresias, São Paulo, Brazil
Energy Localization for Black Holes

- **Energy-momentum localization** has continued to be the subject of interesting works.

- This issue has not yielded a **satisfactory solving**.

- The problem consists in finding a **generally accepted expression** for the energy-momentum density.

- Important results obtained in energy-momentum localization.
The *research work of many scientist* has been focused on this problem.

√ **Used definitions:**

- Superenergy tensors;
- Quasi-local mass definitions;
- Pseudotensorial prescriptions of Einstein, Landau-Lifshitz, Papapetrou, Weinberg (ELLPW), Bergmann-Thomson, Qadir-Sharif and Møller;
- Tele-parallel gravity theory.
Tools for Energy-Momentum Localization

- **Superenergy tensors:**

- **Bel–Robinson (BR) and Bel superenergy tensors** in 1958 and 1959 - 1962, generated from the Weyl and Riemann tensors, and Bel-Debever-Robinson superenergy tensor in 1958.

- **Sachs superenergy tensor** in 1960 Z. Physik 157, 462–477 (1960)

- **Collinson in 1962** Proc. Cambridge Phil. Soc. 58, 346–362 (1962) obtained all four index quadratic in Riemann and divergence-free tensors in four dimensions for any space-time.
• **Properties:** superenergy tensors are closely related to the gravitational energy-momentum.

• **Usefulness:** extended to the algebraic classification of exact solutions to the Einstein equations.

• **Improvements:**


• **Senovilla arXiv:math-ph/0202029** elaborated a general algebraic construction of superenergy tensors from any seed tensor. This superenergy tensor is quadratic in the seed tensor and is characterized by a positivity property that is called the dominant superenergy property (DSEP).

• **Deser showed** *Class. Quant. Grav. 20: L213-L215, 2003:*

\[ \sqrt{\text{In flat space novel conserved symmetric (BR)-like extensions } \ H_{\mu\nu} \ \text{of matter } \ T_{\mu\nu} \ \text{are equivalent to the true stress tensor.}} \]

\[ \sqrt{\text{In curved Ricci-flat backgrounds it is possible to redefine } \ H_{\mu\nu} \ \text{and overcome the non-commutativity of covariant derivatives and preserve conservation.}} \]

√ Positivity and conservation of superenergy tensors, dominant property (DP) for any dimensions and the divergence-free property.

√ Alternative formulation for superenergy tensor using real Clifford algebra that allows to obtain proof of (DSEP) valid for any dimensions.

• Balfagón and Jaén Class.Quant.Grav. 17 (2000) 2491-2498 and http://baldufa.upc.es/xjaen/ttc: studied the four index divergence free quadratic in Riemann tensor polynomials in GR using computational algorithms developed in Mathematica package called Tools of Tensor Calculus (TTC).
• **Quasi-local mass definitions**: any coordinate system


• **Hayward**, Phys. Rev. D49, 831, 1994

• **Hawking** in 1968 proposed a quasi-local energy inside a spacelike closed 2-surface that vanishes when the 2-surface shrinks to a point and it agrees with the asymptotic definition of total energy when the 2-surface becomes very large.

• There is a connection between the Hawking quasi-local energy and the Bel–Robinson tensor, for small spheres it results the relation between the Bel–Robinson tensor and gravitational energy.
• **Berqvist:**

√ **Quasilocal mass definitions yield different results.**

√ **Examples: Reissner-Nordström and Kerr space-times.**


√ **Expression for quasi-local mass evaluated by Tod for Reissner-Nordström space-time.**

√ **Does not give a meaningful result for the Kerr metric.**
• Komar, A. Komar, Phys. Rev. 113, 934 (1959):

√ Defined the mass for asymptotically Minkovski solutions of the Einstein equations.

√ Problem of positivity of mass, Missner pointed out the inconsistency for some geometries.

√ Verified for the RN and Kerr space-times.

√ Available for other metrics:

- Vaydia, Ayón-Beato and Garcia

√ Generalization to other theories of gravity:

- pure Lovelock gravities of all orders
- general Gauss-Bonnet theories
• Energy-momentum complexes

Einstein (E), Landau-Lifshitz (LL), Papapetrou (P), Bergmann-Thomson (B), Weinberg (W) and Møller (M)

√ Structure: - gravitational field
  - matter
  - non-gravitational fields

√ The differential conservation law

\[ \frac{\partial}{\partial x^k} [\sqrt{-g}(T^k_i + t^k_i)] = 0 \]

√ Characteristics - weakness: pseudotensorial quantities, coordinate dependent, excepting the Møller prescription.

√ Rehabilitation: Chang, Nester and Chen, Phys. Rev. Let. 83, 1897, 1999 showed that the energy-momentum complexes are quasi-local and meaningful. Different quasi-local definitions correspond to different boundary conditions.
\textbf{Deser, Franklin, Seminara – Class. Quantum Grav. 16, 2815, 1999:}

\textit{Small regions within matter: the equivalence principle requires an energy-momentum expression dominated by the material energy-momentum tensor.}

\textit{The positivity of gravitational energy is assured by Taylor series expansion in Riemann normal coordinates that is at the second order a positive multiple of the Bel-Robinson tensor.}

\textbf{So, Nester and Chen arXiv:gr-qc/0605150:}

\textit{Demonstrated that for small vacuum regions the ELLPW, B and Goldberg pseudotensors have the zero order material limit required by the equivalence principle, but the Møller does not have.}

\textit{None of these pseudotensors is proportional to the Bel-Robinson tensor.}

\textit{Constructed: an independent combination of B, P and W pseudotensors and one parameter set of linear combination of classical pseudotensors with the desired Bel-Robinson connection.}
• **Powerful concepts for energy-momentum localization:**

• **Motivations:**

- Several energy-momentum complexes "coincide" for any metric of the Kerr-Schild class.

- For many geometries the Einstein, Landau-Lifshitz, Papapetrou, Bergmann-Thomson, Weinberg and Møller prescriptions yield good results.

- For some geometries comply fully with the notion of quasi-local mass introduced by Penrose and elaborated by Tod.

- Different prescriptions yield the same result for a given space-time in Schwarzschild Cartesian coordinates.
This is also available in the case of 2 and 3 dimensional space-times.

In many cases the energy-momentum complexes yield the same results as their tele-parallel versions.

Lessner Gen. Rel. Grav. 28, 527 (1996) sustained in his recent paper that the Møller definition is a powerful concept of energy and momentum in General Relativity.

Cooperstock Mod. Phys. Lett. A14, 1531, 1999 in his very important hypothesis points out that the energy and momentum are confined to the regions of non-vanishing energy-momentum tensor for the matter and all non-gravitational fields.
• The pseudotensorial method has been applied to many black hole solutions:

√ Investigations made by K. S. Virbhadra and J. C. Parikh, followed by A. Chamorro

√ Other investigations performed by Radinschi, Grammenos, Vagenas, I-Ching Yang, Sharif, Gad, Rosen, Sen, Sima, Patashnick, Xulu, Ching-Tzung Yeh, Rue-Ron Hsu, Chin-Rong Lee, Wei-Fui Lin, Halpern, Y.-H. Wei, Matyjasek.

√ Our research is focused on the energy-momentum localization applying several energy-momentum complexes.
Tele-parallel gravity theory

- **Aim:** derive a regularized expression for the gravitational energy-momentum

- **Reformulation:** Einstein’s general relativity can be reformulated in the context of the teleparallel (Weitzenböck) geometry.

- **Tetrad theory of gravitation** first proposed by Møller - the defined energy-momentum complex can be constructed.

- **Teleparallel equivalent of general relativity** \(\text{TEGR}\) - alternative geometrical framework for General Relativity.
• **Important results obtained in tele-parallel gravity theory**: 

√ K. Hayashi, T. Shirafuji 
√ E. W. Mielke 
√ F. W. Hehl 
√ J. D. McCrea 
√ Y. Ne’ eman 
√ W. Kopczyński 
√ S. C. Ulhoa, F. F. Faria
well defined expression for the gravitational energy for asymptotically flat space-times.

extended the expression of the gravitational energy-momentum to an arbitrary set of real-valued tetrad fields.

some results are the same with their pseudotensorial versions
- **Important Results Obtained in Energy-Momentum Localization**

- Meaningful and identical results yielded by different tools.

- **Studied space-times:**

  \( \sqrt{ } \) General nonstatic spherically symmetric metric of the Kerr-Schild class — same result for energy given by ELLPW prescriptions and Penrose quasilocal definition (computed by Tod) — Virbhadra, PRD 60, 104041, 1999

\[
\begin{align*}
  ds^2 &= A(r) du^2 - 2 du dr - r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) \\
  E &= \frac{r}{2} (1 - A(r)) \\
  ds^2 &= dT^2 - dx^2 - dy^2 - dz^2 - (1 - A) \times [dT + \frac{x dx + y dy + z dz}{r}]^2
\end{align*}
\]
Most general nonstatic spherically symmetric metric, Schwarzschild Cartesian coordinates: - Virbhadra, PRD 60, 104041, 1999

$$ds^2 = B(r,t) dt^2 - A(r,t) dr^2 - 2F(r,t) dt dr - D(r,t) r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

- Same results as in the case of the Kerr-Schild Cartesian coordinates obtained using the Einstein prescription (Penrose quasilocal definition computed by Tod) – Schwarzschild, Reissner-Nordström.

- ELLW prescriptions yield the same result for the Einstein-Rosen metric which not of the Kerr-Schild class.

- LL, P and W prescriptions yield different results in Kerr-Schild Cartesian and Schwarzschild Cartesian coordinates.
• Same results and connections obtained in the ELLPW and Møller prescriptions

√ Gamma metric, the energy is equal to the mass m of the black hole.

√ Schwarzschild solution – Einstein and Møller prescriptions yield the same expression for energy, which is equal to the mass m of the black hole.

√ Kerr solution - Einstein and Møller definitions give E=M.

√ Yang and Radinschi, Chin. J. Phys. 42, 40, 2004: a general relation between $\Delta E = E_E - E_M = T_0^0 \times kr^3$, with $T_0^0$ energy density and k=1/2 or k=1.

√ Vagenas, Mod. Phys. Lett. A21, 1947, 2006: a connection between the coefficients of the expression for energy in the E and M prescriptions $\alpha_n^E = 1/(n+1) \alpha_n^M$.

√ Matyjasek Mod. Phys. Lett. A23, 591, 2008 : showed that the Yang and Radinschi and Vagenas hypotheses can be related to the particular distribution of the source for the spherically-symmetric and static systems and find the connection $\Delta E = E_E - E_M = rdE_E/dr$. 
• **Similarity and connections with tele-parallel gravity theory**

✓ General spherically symmetric nonsingular black hole solutions – particular case of Dymnikova space-time, the same result as obtained by Yang and Radinschi using the pseudotensorial prescriptions - Nashed Gamal.

✓ Vacuum non singular black hole in tetrad theory of gravitation – energy content is different from the energy content given by Yang and Radinschi by factor 2 - Nashed Gamal.

✓ Charged axially symmetric solution, energy and angular momentum in tetrad theory of gravitation – good agreement up to $O(a^4)$ with the results obtained using the pseudotensorial prescriptions in the cases of the Kerr-Newman and Bonnor-Vaidya metrics - Nashed Gamal.
• **Energy Localization for Dilatonic Black Holes with a Pure Monopole Field** (I. Radinschi and Th. Grammenos)

• **Black hole solutions under study and their properties:**

- exact black hole solution in a model described by gravity, dilaton field and monopole field,
- its generalization.


\[ \sqrt{\text{Space-times considered are static, spherically symmetric and asymptotically flat.}} \]

\[ \sqrt{\text{Exact solutions are determined from an action that besides gravity contains a dilaton field and a pure (magnetic) monopole field.}} \]
The action is given by
\[ \int d^4x \sqrt{-g} L = \int d^4x \sqrt{-g} \{R - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - (g_1 e^\psi + g_2 e^{-\psi}) F_{\mu\nu} F^{\mu\nu}\} \]

with \( R \) the Ricci scalar, \( \Psi \) represents the dilaton field and \( F_{\mu\nu} \) corresponds to a pure monopole field described by
\[ F = Q \sin \theta \, d\theta \wedge d\phi \]

The solution is established in function of the integration constants \( A, B, \alpha \) and \( \Psi_0 \)
\[ g_1 = \frac{AB}{2Q^2} e^{-\psi_0}, \quad g_2 = \frac{(\alpha-A)(\alpha-B)}{2Q^2} e^{\psi_0}, \quad e^\psi = e^{\psi_0}(1 + \frac{\alpha}{r}) \]
• The metric:

\[ ds^2 = \frac{(r+A)(r+B)}{r(r+\alpha)} dt^2 - \frac{r(r+\alpha)}{(r+A)(r+B)} dr^2 - r(r + \alpha)(d\theta^2 + \sin^2 \theta d\phi^2) \]

• The generalization of this solution → a generalized expression of the action

\[ \int d^4x \sqrt{-g} L = \int d^4x \sqrt{-g} \left\{ R - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - f(\psi) F_{\mu\nu} F^{\mu\nu} \right\} \]

where

\[ f(\psi) = g_1 e^{(c+\sqrt{c^2+1})\psi} + g_2 e^{(c-\sqrt{c^2+1})\psi} \]

• The generalized case is obtained

\[ ds^2 = \frac{(r+A)(r+B)}{r(r+\alpha)} \frac{c}{\sqrt{c^2+1}} dt^2 - \frac{r(r+\alpha)}{(r+A)(r+B)} \frac{r+\alpha}{r} \frac{c}{\sqrt{c^2+1}} dr^2 - \]

\[ -r(r + \alpha) \frac{(r+\alpha)}{r} \frac{c}{\sqrt{c^2+1}} (d\theta^2 + \sin^2 \theta d\phi^2) \]
The Møller energy-momentum complex is given by

\[ \Gamma_i^k = \frac{1}{8\pi} \frac{\partial M_i^{kl}}{\partial x^l}, \]

where

\[ M_i^{kl} = \sqrt{-g} \left( \frac{\partial g_{im}}{\partial x^n} - \frac{\partial g_{in}}{\partial x^m} \right) g^{kn} g^{lm}. \]

The energy and momentum in the Møller prescription are given by

\[ E = \iiint_0^1 \Gamma_0^{0l} dx^1 dx^2 dx^3 = \iiint \frac{\partial M_0^{0l}}{\partial x^i} dx^1 dx^2 dx^3. \]
• **Møller energy-momentum complex for a generalized metric –** \( f(r), f(r)^{-1} \) and \( h(r) \) metric coefficients

- **qload(iri12009);**

- **Calculating ds for th2009 ... Done. (0.000000 sec).**

\[
d s^2 = - \frac{d r^2}{f(r)} - h(r) \ d \theta^2 - h(r) \sin(\theta)^2 \ d \phi^2 + f(r) \ d t^2
\]

- **grcalc(M(dn,up,up));**

- **grdisplay(M(dn,up,up));**

\[
M_{\theta r} = \frac{\sqrt{h(r)^2 \sin(\theta)^2 \ f(r)}}{h(r)} \left( \frac{\partial}{\partial r} h(r) \right)
\]

\[
M_{\phi r} = \frac{\sqrt{h(r)^2 \sin(\theta)^2 \ f(r)}}{h(r)} \left( \frac{\partial}{\partial r} h(r) \right)
\]

\[
M_{t r} = \sqrt{h(r)^2 \sin(\theta)^2 \ f(r)} \left( \frac{\partial}{\partial r} f(r) \right)
\]

\[
M_{\phi \theta} = 2 \frac{\sqrt{h(r)^2 \sin(\theta)^2 \cos(\theta)}}{h(r) \sin(\theta)}
\]
Maple program with the attached GR Tensor platform

- `readlib(grii):`
- `grtENSOR();`
- `grdef(`M{i^k^l}:=sqrt(-detg)*(g{i m,n}-g{in,m})*g{^k^n}*g{^l^m}`):
- Created definition for M(dn,up,up)
- `qload(th2009);`
- Calculating ds for th2009 ... Done. (0.000000 sec).

\[ ds^2 = -\frac{r(r+a) \, dr^2}{(r+A)(r+B)} - r(r+a) \, d\theta^2 - r(r+a) \sin(\theta)^2 \, d\phi^2 + \frac{(r+A)(r+B) \, d\tau^2}{r(r+a)} \]

- `grcalc(M(dn,up,up));`
- `grdisplay(M(dn,up,up));`

\[ M_{\theta \, r} = \sqrt{\frac{r^2(r+a)^2 \sin(\theta)^2 (r+A)(r+B)(2r+a)}{r^2(r+a)^2}} \]
\[ M_{\phi \, r} = \sqrt{\frac{r^2(r+a)^2 \sin(\theta)^2 (r+A)(r+B)(2r+a)}{r^2(r+a)^2}} \]
\[ M_{i \, r} = \sqrt{\frac{r^2(r+a)^2 \sin(\theta)^2 (-r^2B - r^2A - 2rAB + a^2 - aAB)}{r^2(r+a)^2}} \]
\[ M_{\phi \, \theta} = 2\sqrt{\frac{r^2(r+a)^2 \sin(\theta)^2 \cos(\theta)}{r(r+a) \sin(\theta)}} \]
- qload(th42009);
- Calculating ds for th42009 ... Done. (0.000000 sec).

\[ ds^2 = -r(r+a)\left(\frac{r+a}{r}\right)\left(\frac{c}{\sqrt{c^2+1}}\right) \, dr^2 - r(r+a)\left(\frac{r+a}{r}\right)\left(\frac{c}{\sqrt{c^2+1}}\right) \, d\phi^2 \]

\[ -r(r+a)\left(\frac{r+a}{r}\right)\sin(\theta)^2 \, d\phi^2 + \frac{(r+A)(r+B)\left(\frac{r}{r+a}\right)\left(\frac{c}{\sqrt{c^2+1}}\right)}{r(r+a)} \, dt^2 \]

- grcalc(M(dn,up,up));
- grdisplay(M(dn,up,up));

\[ M_{\theta r} = \sqrt{r^2(r+a)^2\left(\frac{c}{\sqrt{c^2+1}}\right)^3 \sin(\theta)^2 \left(\frac{r}{r+a}\right) \left(\frac{c}{\sqrt{c^2+1}}\right)(r+A)(r+B)} \]

\[ \left(\frac{2r\sqrt{c^2+1} - \sqrt{c^2+1} a + c a}{r^2(r+a)^2\left(\frac{c}{\sqrt{c^2+1}}\right)\left(\frac{c}{\sqrt{c^2+1}}\right)} \right) \]
\[ M_\theta^\Phi r = - \sqrt{r^2 (r+a)^2 \left( \frac{c}{\sqrt{c^2+1}} \right)^3 \sin^2 \left( \frac{r}{r+a} \right) (r+A)(r+B) (-2r\sqrt{c^2+1} - \sqrt{c^2+1}a+cA) / \left( r^2 (r+a)^2 \left( \frac{c}{\sqrt{c^2+1}} \right)^2 \sqrt{c^2+1} \right) } \]

\[ M_\theta^t r = \sqrt{r^2 (r+a)^2 \left( \frac{c}{\sqrt{c^2+1}} \right)^3 \sin^2 \left( \frac{r}{r+a} \right) (\sqrt{c^2+1}r^2B - \sqrt{c^2+1}r^2A - 2\sqrt{c^2+1}rAB + \sqrt{c^2+1}ar^2 - \sqrt{c^2+1}arAB + car^2 + carB + carA + caAB) / \left( r^2 (r+a)^2 \left( \frac{c}{\sqrt{c^2+1}} \right)^2 \sqrt{c^2+1} \right) } \]

\[ M_{\phi}^\Phi \theta = 2 \sqrt{r^2 (r+a)^2 \left( \frac{c}{\sqrt{c^2+1}} \right)^3 \sin^2 \left( \frac{r}{r+a} \right) \cos^2 \left( \frac{c}{\sqrt{c^2+1}} \right) \cos(\theta) \left( \frac{c}{\sqrt{c^2+1}} \right) \sin^2 \left( \frac{r}{r+a} \right) \left( r+r+a \right) / \left( r \left( \frac{r+a}{r} \right) \left( \frac{r+a}{r} \right) \sqrt{c^2+1} \right) } \]
• **Energy and Momentum Density Distributions**

• **Results for momentum**

√ All the momenta are found to be zero in the Møller prescription.

• **Results for energy distribution**

√ Møller prescription for the dilatonic black hole with a pure monopole field

\[
E_{\text{init}} = \frac{1}{2} \frac{(-B-A+\alpha)r^2 - 2ABr - \alpha AB}{r(r+\alpha)}
\]

The energy depends on the integration constants \(A\), \(B\) and \(\alpha\), respectively.
Møller prescription for the generalized metric

\[ E_{\text{gen}} = \frac{1}{2} \left[ \sqrt{c^2+1} \left( -r^2 B - r^2 A - 2 r A B + \alpha r^2 - \alpha A B \right) + c (\alpha r^2 + \alpha r B + \alpha r A + \alpha A B) \right] \frac{r(r+\alpha)\sqrt{c^2+1}}{r(r+\alpha)\sqrt{c^2+1}} \]

The connection between the expressions for energy \( E_{\text{init}} \) and \( E_{\text{gen}} \) is

\[ E_{\text{gen}} = E_{\text{init}} + \frac{1}{2} \frac{c}{\sqrt{c^2+1}} \frac{\alpha r^2 + \alpha (A+B)r + \alpha A B}{r(r+\alpha)} \]

The energy distribution depends on the ADM mass, the dilaton field and the monopole charge, in both cases.
### Limit cases and the expressions for energy in each case

<table>
<thead>
<tr>
<th>Limit case</th>
<th>( E_{\text{init}} )</th>
<th>( E_{\text{gen}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r \to \infty )</td>
<td>( \frac{\alpha - (A + B)}{2} )</td>
<td>( \frac{1}{2} \left[ \alpha (1 + \frac{c}{\sqrt{c^2 + 1}}) - (A + B) \right] )</td>
</tr>
<tr>
<td>( r \to 0 )</td>
<td>( \pm \infty )</td>
<td>( \pm \infty )</td>
</tr>
<tr>
<td>( A = B )</td>
<td>( \frac{1}{2} \frac{(-2A + \alpha)r^2 - 2A^2r - \alpha A^2}{r(r + \alpha)} )</td>
<td>( E_{\text{init}} + \frac{1}{2} \frac{c}{\sqrt{c^2 + 1}} \frac{ar^2 + 2A\alpha r + \alpha A^2}{r(r + \alpha)} )</td>
</tr>
<tr>
<td>GHS/GM</td>
<td>( 1 - Ar - 2Q^2 e^{\Psi_0} ) ( \frac{\alpha}{2} )</td>
<td>( E_{\text{init}} + \frac{1}{2} \frac{c}{\sqrt{c^2 + 1}} \frac{2Q^2 e^{\Psi_0} (r + A)}{Ar} )</td>
</tr>
<tr>
<td>( \alpha = B = \frac{2Q^2}{A} e^{\Psi_0} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GHS/GM</td>
<td>( \pm \infty )</td>
<td>( \pm \infty )</td>
</tr>
<tr>
<td>( r \to \infty )</td>
<td>( \frac{-A}{2} )</td>
<td>( \frac{-A}{2} + \frac{1}{A} \frac{c}{\sqrt{c^2 + 1}} \frac{Q^2 e^{\Psi_0}}{Q^2 e^{\Psi_0}} )</td>
</tr>
<tr>
<td>( r \to 0 )</td>
<td>( \pm \infty )</td>
<td>( \pm \infty )</td>
</tr>
</tbody>
</table>
• Conclusions and future work

• Used prescription:

√ Møller.

• Results:

√ All the momenta vanish.

√ For \[ \alpha = B = \frac{2Q^2}{A} e^{\psi_0} \] we obtain the Garfinkle-Horowitz-Strominger / Gibbons-Maeda black hole solution.
General dyonic solution with the energy distribution

\[ g_3 = -\frac{Q_E^2}{Q_M^2} \]

\[ E_{\text{gen}} = E_{\text{init}} + \frac{1}{2} \frac{c}{\sqrt{c^2+1}} \frac{\alpha r^2 + \alpha (A+B)r + \alpha AB}{r(r+\alpha)} \]

Special case of the dyonic solution obtained for \( c=0 \)

\[ E_{\text{init}} = \frac{1}{2} \frac{(-B-A+\alpha)r^2 - 2ABr - \alpha AB}{r(r+\alpha)} \]

Further interpretations are needed for the limit cases.
Thank you for your attention!